

SYMMETRIC TENSOR RANK WITH A TANGENT VECTOR: A GENERIC UNIQUENESS THEOREM

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ABSTRACT. Let $X_{m,d} \subset \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the order d Veronese embedding of \mathbb{P}^m . Let $\tau(X_{m,d}) \subset \mathbb{P}^N$, be the tangent developable of $X_{m,d}$. For each integer $t \geq 2$ let $\tau(X_{m,d}, t) \subseteq \mathbb{P}^N$, be the join of $\tau(X_{m,d})$ and $t-2$ copies of $X_{m,d}$. Here we prove that if $m \geq 2$, $d \geq 7$ and $t \leq 1 + \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$, then for a general $P \in \tau(X_{m,d}, t)$ there are uniquely determined $P_1, \dots, P_{t-2} \in X_{m,d}$ and a unique tangent vector ν of $X_{m,d}$ such that P is in the linear span of $\nu \cup \{P_1, \dots, P_{t-2}\}$, i.e. a degree d linear form f (a symmetric tensor T of order d) associated to P may be written as

$$f = L_{t-1}^{d-1} L_t + \sum_{i=1}^{t-2} L_i^d, \quad (T = v_{t-1}^{\otimes(d-1)} v_t + \sum_{i=1}^{t-2} v_i^{\otimes d})$$

with L_i linear forms on \mathbb{P}^m (v_i vectors over a vector field of dimension $m+1$ respectively), $1 \leq i \leq t$, that are uniquely determined (up to a constant).

1. INTRODUCTION

In this paper we want to address the question of the uniqueness of a particular decomposition for certain given homogeneous polynomials. An analogous question can be rephrased in terms of uniqueness of a particular tensor decomposition of certain given symmetric tensors. In fact, given a homogeneous polynomial f of degree d in $m+1$ variables defined over an algebraically closed field \mathbb{K} , there is an obvious way to associate a symmetric tensor $T \in S^d(V_{\mathbb{K}})$, with $\dim(V_{\mathbb{K}}) = m+1$, to the form f . We will always work over an algebraically closed field \mathbb{K} such that $\text{char}(\mathbb{K}) = 0$. Fix integers $m \geq 2$ and $d \geq 3$. Let $j_{m,d} : \mathbb{P}^m \hookrightarrow \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the order d Veronese embedding of \mathbb{P}^m and set $X_{m,d} := j_{m,d}(\mathbb{P}^m)$ (we often write X instead of $X_{m,d}$). Let $\mathbb{K}[x_0, \dots, x_m]_d$ be the polynomial ring of homogeneous degree d polynomials in $m+1$ variables over \mathbb{K} and let $V_{\mathbb{K}}^*$ be the dual space of $V_{\mathbb{K}}$. Since obviously $\mathbb{P}^m \simeq \mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_1) \simeq \mathbb{P}(V_{\mathbb{K}}^*)$, an element of the Veronese variety $X_{m,d}$ can be interpreted either as the projective class of a d -th power of a linear form $L \in \mathbb{K}[x_0, \dots, x_m]_1$ or as the projective class of a symmetric tensor $T \in S^d(V_{\mathbb{K}}^*) \subset (V_{\mathbb{K}}^*)^{\otimes d}$ for which there exists $v \in V_{\mathbb{K}}^*$ s.t. $T = v^{\otimes d}$. For each integer t such that $1 \leq t \leq N$ let $\sigma_t(X)$ denote the closure in \mathbb{P}^N of the union of all $(t-1)$ -dimensional linear subspaces spanned by t points of X (the t -secant variety of X). From this definition one can understand that the generic element of $\sigma_t(X_{m,d})$ can be interpreted either as $[f] = [L_1^d + \dots + L_t^d] \in$

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$\mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_d)$ with $L_1, \dots, L_t \in \mathbb{K}[x_0, \dots, x_m]_1$ or as $[T] = [v_1^{\otimes d} + \dots + v_t^{\otimes d}] \subset \mathbb{P}(S^d(V_{\mathbb{K}}^*))$ with $v_1, \dots, v_t \in V_{\mathbb{K}}^*$. For a given form f (or a symmetric tensor T), the minimum integer t for which there exists such a decomposition is called the symmetric rank of f (or of T). Finding those v_i 's, $i = 1, \dots, t$ such that $T = v_1^{\otimes d} + \dots + v_t^{\otimes d}$, with t the symmetric rank of T , is known as the Tensor Decomposition problem and it is a generalization of the Singular Value Decomposition problem for symmetric matrices (i.e. if $T \in S^2(V_{\mathbb{K}}^*)$). The existence and the possible uniqueness of the decompositions of a form f as $L_1^d + \dots + L_t^d$ with t minimal is studied in certain cases in [6], [8], [10], [11].

Let $\tau(X) \subseteq \mathbb{P}^N$ be the tangent developable of X , i.e. the closure in \mathbb{P}^N of the union of all embedded tangent spaces $T_P X$, $P \in X$. Obviously $\tau(X) \subseteq \sigma_2(X)$ and $\tau(X)$ is integral. Since $d \geq 3$, the variety $\tau(X)$ is a divisor of $\sigma_2(X)$ ([5], Proposition 3.2). An element in $\tau(X_{m,d})$ can be described both as $[f] \in \mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_d)$ for which there exists two linear forms $L_1, L_2 \in \mathbb{K}[x_0, \dots, x_m]_1$ such that $f = L_1^{d-1}L_2$, and as $[T] \in \mathbb{P}(S^d(V_{\mathbb{K}}^*))$ for which there exists two vectors $v_1, v_2 \in V_{\mathbb{K}}^*$ such that $T = v_1^{\otimes d-1}v_2$ ([5], [4]).

Fix integral positive-dimensional subvarieties $A_1, \dots, A_s \subset \mathbb{P}^N$, $s \geq 2$. The join $[A_1, A_2]$ is the closure in \mathbb{P}^N of the union of all lines spanned by a point of A_1 and a different point of A_2 . If $s \geq 3$ define inductively the join $[A_1, \dots, A_s]$ by the formula $[A_1, \dots, A_s] := [[A_1, \dots, A_{s-1}], A_s]$. The join $[A_1, \dots, A_s]$ is an integral variety and $\dim([A_1, \dots, A_s]) \leq \min\{N, s-1 + \sum_{i=1}^s \dim(A_i)\}$. The integer $\min\{N, s-1 + \sum_{i=1}^s \dim(A_i)\}$ is called the *expected dimension* of the join $[A_1, \dots, A_s]$. Obviously $[A_1, \dots, A_s] = [A_{\sigma(1)}, \dots, A_{\sigma(s)}]$ for any permutation $\sigma : \{1, \dots, s\} \rightarrow \{1, \dots, s\}$. The secant variety $\sigma_t(X)$, $t \geq 2$, is the join of t copies of X . For each integer $t \geq 3$ let $\tau(X, t) \subseteq \mathbb{P}^N$ be the join of $\tau(X)$ and $t-2$ copies of X . We recall that $\min\{N, t(m+1)-2\}$ is the expected dimension of $\tau(X, t)$, while $\min\{N, t(m+1)-1\}$ is the expected dimension of $\sigma_t(X)$. In the range of triples (m, d, t) we will meet in this paper both $\tau(X, t)$ and $\sigma_t(X)$ have the expected dimensions and hence $\tau(X, t)$ is a divisor of $\sigma_t(X)$. An element in $\tau(X_{m,d}, t)$ can be described both as $[f] \in \mathbb{P}(\mathbb{K}[x_0, \dots, x_m]_d)$ for which there exist linear forms $L_1, \dots, L_t \in \mathbb{K}[x_0, \dots, x_m]_1$ such that $f = L_{t-1}^{d-1}L_t + \sum_{i=1}^{t-2} L_i^d$, and as $[T] \in \mathbb{P}(S^d(V_{\mathbb{K}}^*))$ for which there exist $v_1, \dots, v_t \in V_{\mathbb{K}}^*$ such that $T = v_{t-1}^{\otimes(d-1)}v_t + \sum_{i=1}^{t-2} v_i^{\otimes d}$.

After [3], it is natural to ask the following question.

Question 1. Assume $d \geq 3$ and $\tau(X, t) \neq \mathbb{P}^N$. Is a general point of $\tau(X, t)$ in the linear span of a unique set $\{P_0, P_1, \dots, P_{t-2}\}$ with $(P_0, P_1, \dots, P_{t-2}) \in \tau(X) \times X^{t-2}$?

For non weakly $(t-1)$ -degenerate subvarieties of \mathbb{P}^N the corresponding question is true by [8], Proposition 1.5. Here we answer it for a large set of triples of integers (m, d, t) and prove the following result.

Theorem 1. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$. Assume $3 \leq t \leq \beta + 1$. Let P be a general point of $\tau(X, t)$. Then there are uniquely determined points $P_1, \dots, P_{t-2} \in X$ and $Q \in \tau(X)$ such that $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$, i.e. (since $d > 2$) there are uniquely determined points $P_1, \dots, P_{t-2} \in X$ and a unique tangent vector ν of X such that $P \in \langle \{P_1, \dots, P_{t-2}\} \cup \nu \rangle$.*

In terms of homogeneous polynomials Theorem 1 may be rephrased in the following way.

Theorem 2. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$. Assume $3 \leq t \leq \beta + 1$. Let P be a general point of $\tau(X, t)$ and let f be a homogeneous degree d form in $\mathbb{K}[x_0, \dots, x_m]$ associated to P . Then f may be written in a unique way*

$$f = L_{t-1}^{d-1} L_t + \sum_{i=1}^{t-2} L_i^d$$

with $L_i \in \mathbb{K}[x_0, \dots, x_m]_1$, $1 \leq i \leq t$.

In the statement of Theorem 2 the form f is uniquely determined only up to a non-zero scalar, and (as usual in this topic) “uniqueness” may allow not only a permutation of the forms L_1, \dots, L_{t-2} , but also a scalar multiplication of each L_i .

In terms of symmetric tensors Theorem 1 may be rephrased in the following way.

Theorem 3. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$. Assume $3 \leq t \leq \beta + 1$. Let P be a general point of $\tau(X, t)$ and let $T \in S^d(V_{\mathbb{K}}^*)$ be a symmetric tensor associated to P . Then T may be written in a unique way*

$$T = v_{t-1}^{\otimes(d-1)} v_t + \sum_{i=1}^{t-2} v_i^{\otimes d}$$

with $v_i \in V_{\mathbb{K}}^*$, $1 \leq i \leq t$.

As above, in the statement of Theorem 3 the tensor T and the vectors v_i ’s are uniquely determined only up to non-zero scalars.

To prove Theorem 1, and hence Theorems 2 and 3, we adapt the notion and the results on weakly defective varieties described in [6]. It is easy to adapt [6] to joins of different varieties instead of secant varieties of a fixed variety if a general tangent hyperplane is tangent only at one point ([7]). However, a general tangent space of $\tau(X)$ is tangent to $\tau(X)$ along a line, not just at the point of tangency. Hence a general hyperplane tangent to $\tau(X, t)$, $t \geq 3$, is tangent to $\tau(X, t)$ at least along a line. We prove the following result.

Theorem 4. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$. Assume $t \leq \beta + 1$. Let P be a general point of $\tau(X, t)$. Let $P_1, \dots, P_{t-2} \in X$ and $Q \in \tau(X)$ be the points such that $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$. Let ν be the tangent vector of X such that Q is a point of $\langle \nu \rangle \setminus \nu_{red}$. Let $H \subset \mathbb{P}^N$ be a general hyperplane containing the tangent space $T_P \tau(X, t)$ of $\tau(X, t)$. Then H is tangent to X only at the points $P_1, \dots, P_{t-2}, \nu_{red}$, the scheme $H \cap X$ has an ordinary node at each P_i , and H is tangent to $\tau(X) \setminus X$ only along the line $\langle \nu \rangle$.*

2. PRELIMINARIES

Notation 1. Let Y be an integral quasi-projective variety and $Q \in Y_{reg}$. Let $\{kQ, Y\}$ denote the $(k-1)$ -th infinitesimal neighborhood of Q in Y , i.e. the closed subscheme of Y with $(\mathcal{I}_Q)^k$ as its ideal sheaf. If $Y = \mathbb{P}^m$, then we write kQ instead of $\{kQ, \mathbb{P}^m\}$. The scheme $\{kQ, Y\}$ will be called a k -point of Y . We also say that a 2-point is a double point, that a 3-point is a triple point and a 4-point is a quadruple point.

We give here the definition of a $(2, 3)$ -point as it is in [5], p. 977.

Definition 1. Let $\mathfrak{q} \subset \mathbb{K}[x_0, \dots, x_m]$ be the reduced ideal of a simple point $Q \in \mathbb{P}^m$, and let $l \subset \mathbb{K}[x_0, \dots, x_m]$ be the ideal of a reduced line $L \subset \mathbb{P}^m$ through Q . We say that $Z(Q, L)$ is a $(2, 3)$ -point if it is the zero-dimensional scheme whose representative ideal is $(\mathfrak{q}^3 + l^2)$.

Remark 1. Notice that $2Q \subset Z(Q, L) \subset 3Q$.

We recall the notion of weak non-defectivity for an integral and non-degenerate projective variety $Y \subset \mathbb{P}^r$ (see [6]). For any closed subscheme $Z \subset \mathbb{P}^r$ set:

$$(1) \quad \mathcal{H}(-Z) := |\mathcal{I}_{Z, \mathbb{P}^r}(1)|.$$

Notation 2. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $\{2Q, Y\} \subseteq Z$ for all $Q \in Z_{red}$. Fix $H \in \mathcal{H}(-Z)$ where $\mathcal{H}(-Z)$ is defined in (1). Let H_c be the closure in Y of the set of all $Q \in Y_{reg}$ such that $T_Q Y \subseteq H$.

The contact locus H_Z of H is the union of all irreducible components of H_c containing at least one point of Z_{red} .

We use the notation H_Z only in the case $Z_{red} \subset Y_{reg}$.

Fix an integer $k \geq 0$ and assume that $\sigma_{k+1}(Y)$ doesn't fill up the ambient space \mathbb{P}^r . Fix a general $(k+1)$ -uple of points in Y i.e. $(P_0, \dots, P_k) \in Y^{k+1}$ and set

$$(2) \quad Z := \cup_{i=0}^k \{2P_i, Y\}.$$

The following definition of weakly k -defective variety coincides with the one given in [6].

Definition 2. A variety $Y \subset \mathbb{P}^r$ is said to be *weakly k -defective* if $\dim(H_Z) > 0$ for Z as in (2).

In [6], Theorem 1.4, it is proved that if $Y \subset \mathbb{P}^r$ is not weakly k -defective, then $H_Z = Z_{red}$ and that $\text{Sing}(Y \cap H) = (\text{Sing}(Y) \cap H) \cup Z_{red}$ for a general $Z = \cup_{i=0}^k \{2P_i, Y\}$ and a general $H \in \mathcal{H}(-Z)$. Notice that Y is weakly 0-defective if and only if its dual variety $Y^* \subset \mathbb{P}^{r*}$ is not a hypersurface.

In [7] the same authors considered also the case in which Y is not irreducible and hence its joins have as irreducible components the joins of different varieties.

Lemma 1. Fix an integer $y \geq 2$, an integral projective variety Y , $L \in \text{Pic}(Y)$ and $P \in Y_{reg}$. Set $x := \dim(Y)$. Assume $h^0(Y, \mathcal{I}_{(y+1)P} \otimes L) = h^0(Y, L) - \binom{x+y}{x}$. Fix a general $F \in |\mathcal{I}_{yP} \otimes L|$. Then P is an isolated singular point of F .

Proof. Let $u : Y' \rightarrow Y$ denote the blowing-up of Y at P and $E := u^{-1}(P)$ the exceptional divisor. Since $\dim(Y) = x$, we have $E \cong \mathbb{P}^{x-1}$. Set $R := u^*(L)$. For each integer $t \geq 0$ we have $u_*(R(-tE)) \cong \mathcal{I}_{tP} \otimes L$. Thus the push-forward u_* induces an isomorphism between the linear system $|R(-tE)|$ on Y' and the linear system $|\mathcal{I}_{tP} \otimes L|$ on Y . Set $M := R(-yE)$. Since $\mathcal{O}_{Y'}(E)|_E \cong \mathcal{O}_E(-1)$ (up to the identification of E with \mathbb{P}^{x-1}), we have $R(-tE)|_E \cong \mathcal{O}_E(t)$ for all $t \in \mathbb{N}$. Consider on Y' the exact sequence:

$$(3) \quad 0 \rightarrow M(-E) \rightarrow M \rightarrow \mathcal{O}_E(y) \rightarrow 0$$

Our hypothesis implies that $h^0(Y, \mathcal{I}_{yP} \otimes L) = h^0(Y, L) - \binom{x+y-1}{x}$. Thus our assumption implies $h^0(Y', M(-E)) = h^0(Y', R) - \binom{x+y}{x} = h^0(Y', R) - \binom{x+y-1}{x} - \binom{x+y-1}{x-1} = h^0(Y', M) - h^0(E, \mathcal{O}_E(y))$. Thus (3) gives the surjectivity of the restriction map

$\rho : H^0(Y', M) \rightarrow H^0(E, M|_E)$. Since $y \geq 0$, the line bundle $M|_E$ is spanned. Thus the surjectivity of ρ implies that M is spanned at each point of E . Hence M is spanned in a neighborhood of E . Bertini's theorem implies that a general $F' \in |M|$ is smooth in a neighborhood of E . Since F is general and $|M| \cong |\mathcal{I}_{yP} \otimes L|$, P is an isolated singular point of F . \square

3. $\tau(X, t)$ IS NOT WEAK DEFECTIVE

In this section we fix integers $m \geq 2$, $d \geq 3$ and set $N = \binom{m+d}{m} - 1$ and $X := X_{m,d}$. The variety $\tau(X)$ is 0-weakly defective, because a general tangent space of $\tau(X)$ is tangent to $\tau(X)$ along a line. Terracini's lemma for joins implies that a general tangent space of $\tau(X, t)$ is tangent to $\tau(X, t)$ at least along a line (see Remark 3). Thus $\tau(X, t)$ is weakly 0-defective. To handle this problem and prove Theorem 1 we introduce another definition, which is tailor-made to this particular case. As in [5] we want to work with zero-dimensional schemes on X , not on $\tau(X)$ or $\tau(X, t)$. We consider $X = j_{m,d}(\mathbb{P}^m)$ and the 0-dimensional scheme $Z \subset X$ which is the image (via $j_{m,d}$) of the general disjoint union of $t - 2$ double points and one $(2, 3)$ -point of \mathbb{P}^m , in the case of [5] (see Definition 1). We will often work by identifying X with \mathbb{P}^m , so e.g. notice that $\mathcal{H}(-\emptyset)$ is just $|\mathcal{O}_{\mathbb{P}^m}(d)|$.

Remark 2. Fix $P \in X$ and $Q \in T_P X \setminus \{P\}$. Any two such pairs (P, Q) are projectively equivalent for the natural action of $\text{Aut}(\mathbb{P}^m)$. We have $Q \in \tau(X)_{\text{reg}}$ and $T_Q \tau(X) \supset T_P X$. Set $D := \langle \{P, Q\} \rangle$. It is well-known that $D \setminus \{P\}$ is the set of all $O \in \tau(X)_{\text{reg}}$ such that $T_Q \tau(X) = T_O \tau(X)$ (e.g. use that the set of all $g \in \text{Aut}(\mathbb{P}^m)$ fixing P and the line containing P associated to the tangent vector induced by Q acts transitively on $T_P X \setminus D$).

Definition 3. Fix a general $(O_1, \dots, O_{t-2}, O) \in (\mathbb{P}^m)^{t-1}$ and a general line $L \subset \mathbb{P}^m$ such that $O \in L$. Set $Z := Z(O, L) \cup \bigcup_{i=1}^{t-2} 2O_i$. We say that the variety $\tau(X, t)$ is not *drip defective* if $\dim(H_Z) = 0$ for a general $H \in |\mathcal{I}_Z(d)|$.

We are now ready for the following lemma.

Lemma 2. Fix an integer $t \geq 3$ such that $(m+1)t < n$. Let $Z_1 \subset \mathbb{P}^m$ be a general union of a quadruple point and $t - 2$ double points. Let Z_2 be a general union of 2 triple points and $t - 2$ double points. Fix a general disjoint union $Z = Z(O, L) \cup (\bigcup_{i=1}^{t-2} 2P_i)$, where $Z(O, L)$ is a $(2, 3)$ -point as in Definition 1 and O, L and $\{P_1, \dots, P_{t-2}\} \subset \mathbb{P}^m$ are general. Assume $h^1(\mathbb{P}^m, \mathcal{I}_{Z_1}(d)) = h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$. Then:

- (i) $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$;
- (ii) $\tau(X, t)$ is not drip defective;
- (iii) a general $H \in \mathcal{H}(-Z)$ has an ordinary quadratic singularity at each P_i .

Proof. Set $W := 3O \cup (\bigcup_{i=1}^{t-2} 2P_i)$. The definition of a $(2, 3)$ -point gives that $Z(O, L) \subset 3O$. Thus $Z \subset W \subset Z_2$. Hence $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) \leq h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$. Hence part (i) is proven.

To prove part (ii) of the lemma we need to prove that $\dim(H_Z) = 0$ for a general $H \in \mathcal{H}(-Z)$. Since $W \subsetneq Z_1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Z_1}(d)) = 0$, we have $\mathcal{H}(-W) \neq \emptyset$. Since $W_{\text{red}} = Z_{\text{red}}$ and $Z \subset W$, to prove parts (ii) and (iii) of the lemma it is sufficient to prove $\dim((H_W)_c) = 0$ for a general $H_W \in \mathcal{H}(-W)$, where W is as above and $(H_W)_c$ is as in Notation 2. Assume that this is not true, therefore:

- (1) either the contact locus $(H_W)_c$ contains a positive-dimensional component J_i containing some of the P_i 's, for $1 \leq i \leq t-2$,
- (2) or the contact locus $(H_W)_c$ contains a positive-dimensional irreducible component T containing Q .

Set $Z_3 := \cup_{i=1}^{t-3} 2P_i$ and $Z' := 3O \cup Z_3$.

(a) Here we assume the existence of a positive dimensional component $J_i \subset (H_W)_c$ containing one of the P_i 's, say for example $J_{t-2} \ni P_{t-2}$. Thus a general element of $|\mathcal{I}_W(d)|$ is singular along a positive-dimensional irreducible algebraic set containing P_{t-2} . Let $w : M \rightarrow \mathbb{P}^m$ denote the blowing-up of \mathbb{P}^m at the points O, P_1, \dots, P_{t-3} . Set $E_0 := w^{-1}(O)$ and $E_i := w^{-1}(P_i)$, $1 \leq i \leq t-3$. Let A be the only point of M such that $w(A) = P_{t-2}$. For each integer $y \geq 0$ we have $w_*(\mathcal{I}_{yA} \otimes w^*(\mathcal{O}_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \dots - 2E_{t-3})) = \mathcal{I}_{Z' \cup yP_{t-2}}(d)$. Applying Lemma 1 to the variety M , the line bundle $w^*(\mathcal{O}_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \dots - 2E_{t-3})$, the point A and the integer $y = 2$ we get a contradiction.

(b) Here we prove the non-existence of a positive-dimensional $T \subset (H_W)_c$ containing O . Let $w_1 : M_1 \rightarrow \mathbb{P}^m$ denote the blowing-up of \mathbb{P}^m at the points P_1, \dots, P_{t-2} . Set $E_i := w_1^{-1}(P_i)$, $1 \leq i \leq t-2$. Let $B \in M_1$ be the only point of M_1 such that $w_1(B) = O$. For each integer $y \geq 0$ we have $w_{1*}(\mathcal{I}_{yB} \otimes w_1^*(\mathcal{O}_{\mathbb{P}^m}(d))(-2E_1 - \dots - 2E_{t-2})) = \mathcal{I}_{Z' \cup yO}(d)$. Since $h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$ and $|\mathcal{I}_{Z_2}(d)| \subset |\mathcal{I}_Z(d)|$, by Lemma 1 (with $y = 3$) we get a contradiction. \square

In [3], Lemmas 5 and 6, we proved the following two lemmas:

Lemma 3. *Fix integers $m \geq 2$ and $d \geq 5$. If $m \leq 4$, then assume $d \geq 6$. Set $\alpha := \lfloor \binom{m+d-1}{m} / (m+1) \rfloor$. Let $Z_i \subset \mathbb{P}^m$, $i = 1, 2$, be a general union of i triple points and $\alpha - i$ double points. Then $h^1(\mathcal{I}_{Z_i}(d)) = 0$.*

Lemma 4. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$. Let $Z \subset \mathbb{P}^m$ be a general union of one quadruple point and $\beta - 1$ double points. Then $h^i(\mathcal{I}_Z(d)) = 0$.*

We will use the following set-up.

Notation 3. Fix any $Q \in \tau(X) \setminus X$. For $d \geq 3$ the point Q uniquely determines a point $B \in X$ and (up to a non-zero scalar) a tangent vector ν of X with $\nu_{red} = \{B\}$. We have $Q \in \langle \nu \rangle \setminus \{B\}$ and $T_Q \tau(X)$ is tangent to $\tau(X) \setminus X$ exactly along the line $\langle \nu \rangle = \langle \{B, Q\} \rangle$. Let $O \in \mathbb{P}^m$ be the only point such that $j_{n,d}(O) = B$. Let $u_O : \tilde{X} \rightarrow \mathbb{P}^m$ be the blowing-up of O . Let $E := u_O^{-1}(O)$ denote the exceptional divisor. For all integers x, e set $\mathcal{O}_{\tilde{X}}(x, eE) := u^*(\mathcal{O}_{\mathbb{P}^m}(x))(eE)$. Let \mathcal{H} denote the linear system $|\mathcal{O}_{\tilde{X}}(d, -3E)|$ on \tilde{X} .

Remark 3. When $d \geq 4$, the line bundle $\mathcal{O}_{\tilde{X}}(d, -3E)$ is very ample, $u_*(\mathcal{O}_{\tilde{X}}(d, -3E)) = \mathcal{I}_{3O}(1)$, $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(d, -3E)) = \binom{m+d}{m} - \binom{m+2}{m}$ and $h^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(d, -3E)) = 0$ for all $i > 0$.

Lemma 5. *Fix integers $m \geq 2$ and $d \geq 5$. If $m \leq 4$, then assume $d \geq 6$. Set $\alpha := \lfloor \binom{m+d-1}{m} / (m+1) \rfloor$. Fix an integer t such that $3 \leq t \leq \alpha$. The linear system \mathcal{H} on \tilde{X} is not $(t-3)$ -weakly defective. For a general $O_1, \dots, O_{t-2} \in \tilde{X}$ a general $H \in |\mathcal{H}(-2O_1 - \dots - 2O_{t-2})|$ is singular only at the points O_1, \dots, O_{t-2} which are ordinary double points of H .*

Proof. Fix general $O_1, \dots, O_{t-2} \in \tilde{X}$. Fix $j \in \{1, \dots, t-2\}$ and set $Z' := 3O_j \cup \bigcup_{i \neq j} 2O_i$, $Z'' := \bigcup_{i=1}^{t-2} 2O_i$ and $W := 3O_j \cup \bigcup_{i \neq j} 2O_i$. We have $u_*(\mathcal{I}_{Z'}(d, -3E)) \cong \mathcal{I}_{W \cup 3O}(1)$. The case $i = 2$ of Lemma 3 gives $h^1(\mathcal{I}_Z(d, -3E)) = 0$. Lemma 1 applied to a blowing-up of \mathbb{P}^m at $\{O, O_1, \dots, O_{t-2}\} \setminus \{O_j\}$ shows that a general $H \in \mathcal{H}(-Z)$ has as an isolated singular point at O_j . Since this is true for all $j \in \{1, \dots, t-2\}$, \mathcal{H} is not $(t-3)$ -weakly defective (just by the definition of weak defectivity). The second assertion follows from the first one and [6], Theorem 1.4. \square

Now we can apply Lemmas 2, 3, 4 and 5 and get the following result.

Theorem 5. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m} / (m+1) \rfloor$. Fix an integer t such that $3 \leq t \leq \beta + 1$. Then $\tau(X, t)$ is not drip defective.*

Proof. Fix general $P_1, \dots, P_{t-2}, O \in \mathbb{P}^m$ and a general line $L \subset \mathbb{P}^m$ such that $O \in L$. Set $Z := Z(O, L) \cup \bigcup_{i=1}^{t-2} 2P_i$, $W := 3O \cup \bigcup_{i=1}^{t-2} 2P_{t-2}$, $W' := 3O \cup 3O_1 \cup \bigcup_{i=2}^{t-2} 2P_{t-2}$ and $W'' := 4O \cup \bigcup_{i=1}^{t-2} 2P_{t-2}$. Take $O_i \in \tilde{X}$ such that $u_O(O_i) = P_i$, $1 \leq i \leq t-2$. Since $u_{O_*}(\mathcal{I}_{2O_1 \cup \dots \cup 2O_{t-2}}(d, -4E)) \cong \mathcal{I}_W(d)$, Lemma 4 gives $h^1(\mathcal{I}_{2O_1 \cup \dots \cup 2O_{t-2}}(d, -4E)) = 0$. Since $Z(O, L) \subset 3O$, the case $y = 3$ of Lemma 1 applied to the blowing-up of \mathbb{P}^m at O_1, \dots, O_{t-2} shows that a general $H \in |\mathcal{I}_W(d)|$ has an isolated singularity at O with multiplicity at most 3. \square

Recall that $\text{Sing}(\tau(X)) = X$ and that for each $Q \in \tau(X) \setminus X$ there is a unique $O \in X$ and a unique tangent vector ν to X at O such that $Q \in \langle \nu \rangle$ and that $\langle \nu \rangle \setminus \{O\}$ is the contact locus of the tangent space $T_Q \tau(X)$ with $\tau(X) \setminus X$.

Let P be a general point of $\tau(X, t)$, i.e. fix a general $(P_1, \dots, P_{t-2}, Q) \in X^{t-2} \times \tau(X)$ and a general $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$.

Proof of Theorem 1. Fix a general $P \in \tau(X, t)$, say $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$ with (P_1, \dots, P_{t-2}, Q) general in $X^{t-2} \times \tau(X)$. Terracini's lemma for joins ([1], Corollary 1.10) gives $T_P \tau(X, t) = \langle T_{P_1} X \cup \dots \cup T_{P_{t-2}} X \cup T_Q \tau(X) \rangle$. Let O be the point of \mathbb{P}^m such that $Q \in T_{j_{m,d}(O)} X$. Let \mathcal{H}' (resp. \mathcal{H}'') be the set of all hyperplane $H \subset \mathbb{P}^N$ containing $T_Q \tau(X)$ (resp. $T_P \tau(X, t)$). We may see \mathcal{H}' and \mathcal{H}'' as linear systems on the blowing-up \tilde{X} of \mathbb{P}^m at O . Take $O_i \in \tilde{X}$, $1 \leq i \leq t-2$, such that $P_i = u(O_i)$ for all i . We have $\mathcal{H}'' = \mathcal{H}'(-2P_1 - \dots - 2P_{t-2})$ and $\mathcal{H} \subseteq \mathcal{H}'$, where \mathcal{H} is defined in Notation 3. Since (P_1, \dots, P_{t-2}) is general in X^{t-2} for a fixed Q and $\mathcal{H} \subseteq \mathcal{H}'$, Lemma 5 gives that a general $H \in \mathcal{H}''$ intersects X in a divisor which, outside O , is singular only at P_1, \dots, P_{t-2} and with an ordinary node at each P_i . Now assume $P \in \langle \{P'_1, \dots, P'_{t-2}, Q'\} \rangle$ for some other $(P'_1, \dots, P'_{t-2}, Q') \in X^{t-2} \times \tau(X)$. Since P is general in $\tau(X, t)$ and $\tau(X, t)$ has the expected dimension, the $(t-1)$ -ple $(P'_1, \dots, P'_{t-2}, Q')$ is general in $X^{t-2} \times \tau(X)$. Hence $H \cap X$ is singular at each P'_i , $1 \leq i \leq t-2$, and with an ordinary node at each P'_i . Since O is not an ordinary node of $H \cap X$, we get $\{P_1, \dots, P_{t-2}\} = \{P'_1, \dots, P'_{t-2}\}$. Thus $O = O'$. Hence H is tangent to $\tau(X)_{reg}$ exactly along the line $\langle \{Q, O\} \rangle \setminus \{O\}$. Hence $Q' \in \langle \{Q, O\} \rangle$. Assume $Q \neq Q'$. Since P is general in $\tau(X, t)$, then $P \notin \tau(X, t-1)$. Hence $Q' \notin \langle \{P_1, \dots, P_{t-2}\} \rangle$ and $Q \notin \langle \{P_1, \dots, P_{t-2}\} \rangle$. Thus $\langle \{P_1, \dots, P_{t-2}, Q\} \rangle \cap \langle \{P_1, \dots, P_{t-2}, Q'\} \rangle = \langle \{P_1, \dots, P_{t-2}\} \rangle$ if $Q \neq Q'$. Since $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle \cap \langle \{P_1, \dots, P_{t-2}, Q'\} \rangle$, we got a contradiction. \square

Proof of Theorem 4. The case $t = 2$ is well-known and follows from the following fact: for any $O \in X$ and any $Q \in T_O X \setminus \{O\}$ the group $G_O := \{g \in$

$\text{Aut}(\mathbb{P}^n) : g(O) = O\}$ acts on $T_O X$ and the stabilizer $G_{O,Q}$ of Q for this action is the line $\langle\{O, Q\}\rangle$, while $T_O X \setminus \langle\{O, Q\}\rangle$ is another orbit for $G_{O,Q}$. Thus we may assume $t \geq 3$. Fix a general $P \in \tau(X, t)$ and a general hyperplane $H \supset T_P \tau(X, t)$. If H is tangent to $\tau(X)$ at a point $Q' \in \tau(X) \setminus X$, then it is tangent along a line containing Q' . Let $E \in X$ be the only point such that $Q' \in T_E X$. We get $T_E X \subset \tau(X, t)$ and that $H \cap T_E X$ is larger than the double point $2E \subset X$. Theorem 1 gives that Q, Q' and E are collinear, i.e H is tangent only along the line ν . \square

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